

Quantization of Gaussian measures with Rényi- α -entropy constraints

Wolfgang Kreitmeier *

Abstract

We consider the optimal quantization problem with Rényi- α -entropy constraints for centered Gaussian measures on a separable Banach space. For $\alpha = \infty$ we can compute the optimal quantization error by a moment on a ball. For $\alpha \in]1, \infty]$ and large entropy bound we derive sharp asymptotics for the optimal quantization error in terms of the small ball probability of the Gaussian measure. We apply our results to several classes of Gaussian measures. The asymptotical order of the optimal quantization error for $\alpha > 1$ is different from the well-known cases $\alpha = 0$ and $\alpha = 1$.

Keywords Gaussian measures, Rényi- α -entropy, functional quantization, high-resolution quantization.

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1 Introduction and basic notation

Let $\mathbb{N} := \{1, 2, \dots\}$. Let $\alpha \in [0, \infty]$ and $p = (p_1, p_2, \dots) \in [0, 1]^{\mathbb{N}}$ be a probability vector, i.e. $\sum_{i=1}^{\infty} p_i = 1$. The Rényi- α -entropy $\hat{H}^{\alpha}(p) \in [0, \infty]$ is defined as (see e.g. [1, Definition 5.2.35] resp. [4, Chapter 1.2.1])

$$\hat{H}^{\alpha}(p) = \begin{cases} -\sum_{i=1}^{\infty} p_i \log(p_i), & \text{if } \alpha = 1 \\ -\log(\max\{p_i : i \in \mathbb{N}\}), & \text{if } \alpha = \infty \\ \frac{1}{1-\alpha} \log(\sum_{i=1}^{\infty} p_i^{\alpha}), & \text{if } \alpha \in [0, \infty[\setminus \{1\}. \end{cases}$$

We use the convention $0 \cdot \log(0) := 0$ and $0^x := 0$ for all real x . The logarithm \log is based on e .

*Department of Informatics and Mathematics,
University of Passau, 94032 Passau, Germany
e-mail: wolfgang.kreitmeier@uni-passau.de
phone: +49(0)851/509-3014

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Remark 1.1. *With these conventions we obtain*

$$\hat{H}^0(p) = \log(\text{card}\{p_i : i \in \mathbb{N}, p_i > 0\}),$$

if card denotes cardinality. Using the rule of de l'Hospital it is easy to see, that

$$\lim_{\substack{\alpha \rightarrow 1 \\ \alpha \neq 1}} \hat{H}^\alpha(\cdot) = \hat{H}^1(\cdot)$$

(cf. [1, Remark 5.2.34]). Moreover, $\lim_{\alpha \rightarrow \infty} \hat{H}^\alpha(\cdot) = \hat{H}^\infty(\cdot)$.

Let $(E, \|\cdot\|)$ be a real separable Banach space with norm $\|\cdot\|$. Let μ be a Borel probability measure on E . Denote by \mathcal{F} the set of all Borel-measurable mappings $f : E \rightarrow E$ with $\text{card}(f(E)) \leq \text{card}(\mathbb{N})$. A mapping $f \in \mathcal{F}$ is called **quantizer** and the image $f(E)$ is called codebook consisting of codepoints. We assume throughout the whole paper that the codepoints are distinct. Every quantizer f induces a partition $\{f^{-1}(z) : z \in f(E)\}$ of E . Every element of this partition is called **codecell**. The image measure $\mu \circ f^{-1}$ has a countable support and defines an approximation of μ , the so-called quantization of μ by f . For any enumeration $\{z_1, z_2, \dots\}$ of $f(E)$ we define

$$H_\mu^\alpha(f) = \hat{H}^\alpha((\mu \circ f^{-1}(z_1), \mu \circ f^{-1}(z_2), \dots))$$

as the Rényi- α -entropy of f w.r.t μ . Now we intend to quantify the distance between μ and its approximation under f . To this end let $\rho : [0, \infty[\rightarrow [0, \infty[$ be a surjective, strictly increasing and continuous mapping with $\rho(0) = 0$. Hence ρ is invertible. The inverse function is denoted by ρ^{-1} and also strictly increasing. We assume throughout the whole paper that $\int \rho(\|x\|) d\mu(x) < \infty$. For $f \in \mathcal{F}$ we define as distance between μ and $\mu \circ f^{-1}$ the quantization error

$$D_{\mu, \rho}(f) = \int \rho(\|x - f(x)\|) d\mu(x).$$

For any $R \geq 0$ we denote by

$$D_{\mu, \rho}^\alpha(R) = \inf\{D_{\mu, \rho}(f) : f \in \mathcal{F}, H_\mu^\alpha(f) \leq R\} \quad (1)$$

the optimal quantization error for μ under Rényi- α -entropy bound R . Indeed, it is justified to speak of a distance. It was shown by the author [31] in the finite-dimensional case and for Euclidean norm that for a large class of distributions μ the optimal quantization error (1) is equal to a Wasserstein distance.

Remark 1.2. *The optimal quantization error is decreasing in $\alpha \geq 0$. To see this let $f \in \mathcal{F}$ with $H_\mu^\alpha(f) \leq R$. For arbitrary $0 < \gamma \leq \beta < \infty$ we have (cf. [3], p. 53)*

$$H_\mu^\beta(f) \leq H_\mu^\gamma(f).$$

Together with Remark 1.1 we conclude that $H_\mu^\alpha(f) \leq H_\mu^0(f)$. In view of Definition (1) we thus obtain

$$D_{\mu, \rho}^\alpha(R) \leq \inf\{D_{\mu, \rho}(g) : g \in \mathcal{F}, H_\mu^0(g) \leq R\} = D_{\mu, \rho}^0(R).$$

An exact determination of the optimal quantization error (1) for every $R \geq 0$ was successfully only in a few special cases so far. In this regard most is known in the one-dimensional case under the restriction

$$\rho(x) = x^r \text{ with } r > 0 \quad (2)$$

and $\alpha \in \{0, 1\}$. In case of $\alpha = 0$ the reader is referred to [24, section 5.2]. To the author's knowledge the uniform and the exponential distribution are the only examples for $\alpha = 1$ where an exact determination of the optimal quantization error was carried out so far. György and Linder [28] have determined a parametric representation of (1) for the uniform distribution and a large class of distance functions ρ which includes (2). Berger [6] has derived in case of $\alpha = 1$ and $r = 2$ an analytical representation for the optimal quantization error of the exponential distribution. For the class (2) of distance functions the author [29] was able to generalize the results of György and Linder [28] to the case $\alpha \in [0, \infty]$.

Due to the difficulties in determining the optimal quantization error one is interested in asymptotics for the error for large entropy bounds. In case of $\alpha \in \{0, 1\}$ and finite dimension the asymptotical behaviour of the optimal quantization error is well-known for a large class of distributions, see e.g. [24, 27]. Kreitmeier and Linder [32] have derived also sharp asymptotics for a large class of one-dimensional distributions and $\alpha \in [0, \infty]$. Moreover, the author [30] has determined first-order asymptotics for the optimal quantization error in arbitrary finite dimension and $\alpha \in [0, \infty]$, where the class of distributions is larger than the one in [32].

This paper aims to determine asymptotics for the optimal quantization error (1) in the infinite dimensional case. To this end we will assume for the rest of this paper that $(E, \|\cdot\|)$ is of infinite dimension. Moreover, we restrict ourselves to Gaussian measures. In more detail we will assume from now on that μ is a non-atomic centered Gaussian measure on E and the support of μ coincides with E . The restriction to Gaussian measures is motivated by different reasons. First, this class of distributions has been extensively studied in the past. In the proofs of this paper we especially use concentration inequalities (cf. [8]) and small ball asymptotics (see e.g. [5, 9, 10, 21, 34, 39, 43, 44]). Secondly, for distance functions of type (2) and $\alpha \in \{0, 1\}$ the asymptotical order of $D_{\mu, \rho}^\alpha(R)$ for large R has been already determined for several classes of Gaussian measures. Dereich et al. [13] have determined asymptotics for (1) in case of $\alpha = 0$ and for distance functions of type (2). Their results require weak conditions on the regular variation of the small ball asymptotics of the Gaussian measure. Graf, Luschgy and Pagès [25] have additionally shown for $\alpha = 0$ and restriction (2) that one can determine the small ball asymptotics from the asymptotics of the optimal quantization error (1) if the asymptotics of (1) satisfy certain regularity conditions. Luschgy and Pagès [36] have determined sharp error asymptotics for $\alpha = 0$ and distance function $\rho(x) = x^2$. They imposed a condition on the regularity of the eigenvalues of the covariance operator of μ . In this situation, also the sharp error asymptotics for $\alpha = 0$ and $\alpha = 1$ coincide,

cf. [26]. Dereich and Scheutzow [14] have shown for fractional Brownian motion that sharp asymptotics of (1) for large R exist and also coincide for $\alpha \in \{0, 1\}$. According to these cited works the asymptotics for $\alpha = 0$ and $\alpha = 1$ are of the same order and in view of Remark 1.2 even for all $\alpha \in [0, 1]$.

The objective of this paper is to analyze the optimal quantization error for $\alpha > 1$. In Section 2 we determine in case of $\alpha = \infty$ ('mass-constrained quantization') a representation of the optimal quantization error by a moment on a ball (cf. Proposition 2.3). The proof of this result is a straightforward generalization of the techniques used in the proof of [30, Proposition 2.1.]. In Section 3, for a large class of Gaussian measures where sharp asymptotics for the small ball probability are known, we can determine sharp asymptotics for the optimal quantization error with entropy parameter $\alpha > 1$ (cf. Corollary 3.12, Theorem 3.15). The cornerstone of our approach is covered by Proposition 3.10. For distance functions of type (2) we obtain a representation of the sharp asymptotics for $D_{\mu, \rho}^\infty$ in terms of the inverse of the small ball function (cf. definition (6)). The condition imposed (cf. (10)) on the small ball asymptotics is satisfied by most prevalent Gaussian measures. For those distributions we are then able to derive also sharp asymptotics for all $\alpha > 1$, cf. Corollary 3.16. In Section 4 we discuss several examples of Gaussian processes in order to determine the asymptotical order of the optimal quantization error for large entropy bound and $\alpha > 1$. The asymptotics of the optimal quantization error for $\alpha > 1$ turns out to be of different order compared to the case $\alpha \leq 1$.

2 The optimal quantization error under mass-constraints

Let $f \in \mathcal{F}$ and $R > 0$ with $H_\mu^\infty(f) \leq R$. From the definition we obtain

$$\max\{\mu(f^{-1}(a)) : a \in f(E)\} \geq e^{-R}.$$

Hence we call optimal quantization with $\alpha = \infty$ mass-constrained quantization. Denote by \mathbb{R} all real numbers, let

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} \text{ and } \mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}.$$

As a key tool we will use Anderson's inequality [2] as stated in reference [8].

Theorem 2.1 ([8, Corollary 4.2.3]). *If A is a convex, symmetric and Borel-measurable subset of E , then for every $a \in E$*

$$\mu(A) \geq \mu(A + a).$$

Moreover, the function

$$\mathbb{R} \ni t \rightarrow \int_E g(x + ta) \mu(dx)$$

is nondecreasing on \mathbb{R}_0^+ , provided $g : E \rightarrow \mathbb{R}$ is such that the sets $\{g \leq c\}, c \in \mathbb{R}$, are symmetric and convex, and $g(\cdot + ta)$ is μ -integrable for any $t \geq 0$.

We denote by $\text{supp}(\mu)$ the support of μ . For $a \in E$ and $s > 0$ we denote by

$$B(a, s) = \{x \in E : \|x - a\| \leq s\}$$

the closed ball around a with radius s . We deduce from [8, Corollary 4.4.2 (i)] that the mapping

$$\mathbb{R}^+ \ni t \xrightarrow{F_3} \mu(B(a, t)) \in \mathbb{R}^+ \quad (3)$$

is continuous. Because μ is non-atomic the mapping F_a has a continuous extension to \mathbb{R}_0^+ which we call also F_a and $F_a(0) = 0$. For any set $A \subset E$ we denote by 1_A the characteristic function on A .

Lemma 2.2. *Let $a \in E$ and $A \subset E$ be a Borel measurable set with $\mu(A) \in]0, 1[$. Then there exists an $s \in]0, \infty[$ such that $\mu(A) = \mu(B(0, s))$ and*

$$\int_A \rho(\|x - a\|) d\mu(x) \geq \int_{B(0, s)} \rho(\|x\|) d\mu(x).$$

Proof.

1. $\int_A \rho(\|x - a\|) d\mu(x) \geq \int_{B(a, l)} \rho(\|x - a\|) d\mu(x)$ with $\mu(B(a, l)) = \mu(A)$.

By the properties of the mapping F_a an $l > 0$ exists with $\mu(B(a, l)) = \mu(A)$. The remaining part of the proof can be taken from the proof of [24, Lemma 2.8]. Although [24, Lemma 2.8] covers only the special case $\rho(x) = x^r$, the argument works also for general ρ .

2. $\int_{B(a, l)} \rho(\|x - a\|) d\mu(x) \geq \int_{B(0, s)} \rho(\|x\|) d\mu(x)$.

Let

$$E \ni x \rightarrow f(x) = 1_{B(0, s)}(x) \rho(\|x\|).$$

By Theorem 2.1 an $s > 0$ exists such that

$$\mu(B(0, s)) = \mu(B(a, l)) \leq \mu(B(0, l)),$$

which yields $s \leq l$. For every $c \in [0, \infty[$ the set

$$B(0, s) \cap \{f \leq c\} = \{x \in B(0, s) : \rho(\|x\|) \leq c\} = B(0, \min(s, \rho^{-1}(c)))$$

is symmetric and convex. Moreover $f(\cdot + t(-a))$ is μ -integrable for every $t \geq 0$. Because the support of f is a subset of $B(0, s)$ we obtain from Theorem 2.1 that

$$\begin{aligned} \int_{B(a, l)} \rho(\|x - a\|) d\mu(x) &\geq \int_{B(a, s)} \rho(\|x - a\|) d\mu(x) \\ &= \int f(x - a) d\mu(x) \geq \int f(x) d\mu(x) \\ &= \int_{B(0, s)} \rho(\|x\|) d\mu(x). \end{aligned}$$

□

The proof of the following Proposition 2.3 is an obvious generalization of the proof of [30, Proposition 2.1.] to the finite-dimensional case. The main idea of constructing a quantizer based on a countable partition of E works also for infinite dimensional separable Banach spaces. For the reader's convenience we provide a complete proof.

Proposition 2.3. *Let $R > 0$ and $s > 0$ such that $\mu(B(0, s)) = e^{-R}$. Then*

$$D_{\mu, \rho}^{\infty}(R) = \int_{B(0, s)} \rho(\|x\|) d\mu(x). \quad (4)$$

Proof. Let $R > 0$. From the definition (1) of $D_{\mu, \rho}^{\infty}(R)$ we obtain

$$D_{\mu, \rho}^{\infty}(R) = \inf \left\{ \int \rho(\|x - f(x)\|) d\mu(x) : f \in \mathcal{F}, \max_{a \in f(E)} \mu(f^{-1}(a)) \geq e^{-R} \right\}$$

Moreover let

$$D(R) = \inf \left\{ \int_A \rho(\|x - a\|) d\mu(x) : a \in E, A \text{ measurable}, \mu(A) \geq e^{-R} \right\}. \quad (5)$$

1. $D_{\mu, \rho}^{\infty}(R) \geq D(R)$.

Let $f \in \mathcal{F}$ with $\max_{b \in f(E)} \mu(f^{-1}(b)) \geq e^{-R}$. Then an $a \in f(E)$ exists with $\mu(f^{-1}(a)) \geq e^{-R}$. Let $A = f^{-1}(a)$. We obtain

$$\begin{aligned} \int \rho(\|x - f(x)\|) d\mu(x) &= \sum_{b \in f(E)} \int_{f^{-1}(b)} \rho(\|x - b\|) d\mu(x) \\ &\geq \int_{f^{-1}(a)} \rho(\|x - a\|) d\mu(x) \\ &= \int_A \rho(\|x - a\|) d\mu(x) \geq D(R), \end{aligned}$$

which yields $D_{\mu, \rho}^{\infty}(R) \geq D(R)$.

2. $D_{\mu, \rho}^{\infty}(R) \leq D(R)$.

Let $A \subset E$ be measurable with $\mu(A) \geq e^{-R}$ and choose $a \in E$. Let $\varepsilon > 0$. Because $\rho(0) = 0$ and ρ is continuous, a $\delta > 0$ exists such that for every $t \in [0, \delta]$ we have $\rho(t) \leq \varepsilon$. Let $(x_n)_{n \in \mathbb{N}}$ be dense in E . Then $(B(x_n, \delta))_{n \in \mathbb{N}}$ is an open cover of E . Hence a Borel-measurable partition $(A_n)_{n \in \mathbb{N}}$ of $E \setminus A$ exists, such that $A_n \subset B(x_n, \delta)$ for every $n \in \mathbb{N}$. Now we define the mapping $f : E \rightarrow E$ by

$$f(x) = \begin{cases} a, & \text{if } x \in A \\ x_n, & \text{if } x \in A_n. \end{cases}$$

Obviously $f \in \mathcal{F}$ and $\max_{b \in f(E)} \mu(f^{-1}(b)) \geq \mu(A) \geq e^{-R}$. We deduce

$$\begin{aligned}
D_{\mu,\rho}^\infty(R) &\leq \int_E \rho(\|x - f(x)\|) d\mu(x) \\
&= \int_A \rho(\|x - a\|) d\mu(x) + \sum_{n \in \mathbb{N}} \int_{A_n} \rho(\|x - x_n\|) d\mu(x) \\
&\leq \int_A \rho(\|x - a\|) d\mu(x) + \sum_{n \in \mathbb{N}} \varepsilon \mu(A_n) \\
&= \int_A \rho(\|x - a\|) d\mu(x) + \varepsilon.
\end{aligned}$$

Because $\varepsilon > 0$, $a \in E$ and the set $A \subset E$ were arbitrary we obtain that $D_{\mu,\rho}^\alpha(R) \leq D(R)$.

3. Proof of equation (4).

From step 1 and 2 we deduce $D_{\mu,\rho}^\infty(R) = D(R)$. Obviously we can assume that $\mu(A) \in]0, 1[$ for the set A in (5). But then the assertion follows from Lemma 2.2. \square

For $s > 0$ let

$$b_\mu(s) = -\log(\mu(\{x \in E : \|x\| \leq s\})) = -\log(\mu(B(0, s))) \quad (6)$$

be the small ball function of μ . Note that $b_\mu(\cdot)$ is continuous, surjective, strictly decreasing and, therefore, invertible (see e.g. [12, Lemma 2.3.5]). Thus we obtain as an immediate consequence of Proposition 2.3 the following result.

Corollary 2.4. *Let $R > 0$. Then $D_{\mu,\rho}^\infty(R) \leq \rho(b_\mu^{-1}(R))e^{-R}$.*

3 High-rate error asymptotics

In this section we will prove high-rate error asymptotics for the optimal quantization error with entropy index $\alpha \in]1, \infty]$. If the small ball function $b_\mu(\cdot)$ has a certain asymptotical behavior we can determine the sharp asymptotics of the optimal quantization error for large entropy bound (cf. Corollary 3.12, Theorem 3.15). We begin with an upper bound for the optimal quantization error. As with Proposition 2.3 the proof of the following result is a straightforward generalization of the proof of [30, Proposition 2.1].

Lemma 3.1. *Let $\alpha > 1$ and $R > 0$. Then*

$$\begin{aligned}
D_{\mu,\rho}^\alpha(R) &\leq \inf \left\{ \int_{B(a,s)} \rho(\|x - a\|) d\mu(x) : a \in E, s > 0, \mu(B(a, s)) \geq e^{-\frac{\alpha-1}{\alpha}R} \right\} \\
&= D_{\mu,\rho}^\infty \left(\frac{\alpha-1}{\alpha} R \right).
\end{aligned}$$

Proof. The second part of the assertion is an immediate consequence of Proposition 2.3 and Lemma 2.2. To prove the first part let $a \in E$ and $s > 0$ with $\mu(B(a, s)) \geq e^{-\frac{\alpha-1}{\alpha}R}$. Let $(a_n)_{n \in \mathbb{N}}$ be a dense subset of E . Let $\varepsilon > 0$. Because $\rho(0) = 0$ and ρ is continuous, a $\delta > 0$ exists such that $\rho(t) < \varepsilon$ for every $t \in [0, \delta]$. Hence $(B(a_n, \delta))_{n \in \mathbb{N}}$ is an open cover of E . Thus a Borel-measurable partition $(A_n)_{n \in \mathbb{N}}$ of $E \setminus B(a, s)$ exists, with $A_n \subset B(a_n, \delta)$ for every $n \in \mathbb{N}$. We define the mapping $f : E \rightarrow E$ by

$$f(x) = \begin{cases} a, & \text{if } x \in B(a, s) \\ a_n, & \text{if } x \in A_n. \end{cases}$$

Obviously $f \in \mathcal{F}$. Due to $\alpha > 1$ we obtain

$$\begin{aligned} H_\mu^\alpha(f) &= \frac{1}{1-\alpha} \log \left(\mu(B(a, s))^\alpha + \sum_{n=1}^{\infty} \mu(A_n)^\alpha \right) \\ &\leq \frac{1}{1-\alpha} \log(\mu(B(a, s))^\alpha) \leq \frac{1}{1-\alpha} \log(e^{(1-\alpha)R}) = R. \end{aligned}$$

As a consequence we get

$$\begin{aligned} D_{\mu, \rho}^\alpha(R) \leq D_{\mu, \rho}(f) &= \int_{B(a, s)} \rho(\|x - a\|) d\mu(x) + \sum_{n=1}^{\infty} \int_{A_n} \rho(\|x - a_n\|) d\mu(x) \\ &\leq \int_{B(a, s)} \rho(\|x - a\|) d\mu(x) + \sum_{n=1}^{\infty} \mu(A_n) \varepsilon \\ &\leq \int_{B(a, s)} \rho(\|x - a\|) d\mu(x) + \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily the assertion is proved. \square

Let $r > 0$. From now on we will assume for the rest of this paper that $\rho(x) = x^r$ for every $x \in \mathbb{R}_0^+$. To stress this choice for ρ we write $D_{\mu, r}^\alpha(\cdot)$ instead of $D_{\mu, \rho}^\alpha(\cdot)$.

Remark 3.2. *The r -th moment is always finite for Gaussian measures. This can be deduced either from Fernique's theorem (cf. [8, Theorem 2.8.5]) or follows from concentration inequalities for Gaussian measures (see e.g. [8, Theorem 4.3.3] or [12, p. 25]).*

In order to formulate rates we introduce the following notations. For mappings $f, g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ and $a \in [0, \infty]$ we write $f \sim g$ as $x \rightarrow a$ if $\lim_{x \rightarrow a} f(x)/g(x) = 1$. We denote $f \gtrsim g$ as $x \rightarrow a$ if

$$0 < \liminf_{x \rightarrow a} f(x)/g(x)$$

and $f \gtrsim g$ as $x \rightarrow a$ if

$$1 \leq \liminf_{x \rightarrow a} f(x)/g(x).$$

We write $f \lesssim g$ as $x \rightarrow a$ if

$$\limsup_{x \rightarrow a} f(x)/g(x) < \infty$$

and $f \lesssim g$ as $x \rightarrow a$ if

$$\limsup_{x \rightarrow a} f(x)/g(x) \leq 1.$$

If $f \gtrsim g$ and $f \lesssim g$ we write $f \approx g$. Obviously $f \sim g$, if $f \gtrsim g$ and $f \lesssim g$.

The following Lemma has been proved by Dereich (cf. [13, Lemma 4.2]). See also the proof of Corollary 1.3 in [25].

Lemma 3.3. *Let $c > 0$. Let $a > 0$ and $b \in \mathbb{R}$. If*

$$b_\mu(s) \sim c(1/s)^a (\log(1/s))^b \text{ as } s \rightarrow 0,$$

then

$$b_\mu^{-1}(R) \sim c^{1/a} a^{-b/a} R^{-1/a} (\log(R))^{b/a} \text{ as } R \rightarrow \infty.$$

Remark 3.4. *It is also easy to check that*

$$b_\mu(s) \approx (1/s)^a (\log(1/s))^b \text{ as } s \rightarrow 0$$

implies that

$$b_\mu^{-1}(R) \approx R^{-1/a} (\log(R))^{b/a} \text{ as } R \rightarrow \infty.$$

Remark 3.5. *According to the separability of the support of μ and the finite r -th moment (cf. Remark 3.2) we have*

$$\lim_{R \rightarrow \infty} D_{\mu,r}^0(R) = 0.$$

In view of Remark 1.2 we thus get

$$\lim_{R \rightarrow \infty} D_{\mu,r}^\alpha(R) = 0$$

for every $\alpha \in [0, \infty]$.

Definition 3.6. *A family $(f_R)_{R>0} \subset \mathcal{F}$ of quantizers is called asymptotically α -optimal, if $H_\mu^\alpha(f_R) \leq R$ for every $R > 0$ and*

$$\lim_{R \rightarrow \infty} \frac{D_{\mu,r}(f_R)}{D_{\mu,r}^\alpha(R)} = 1. \quad (7)$$

Remark 3.7. *An asymptotically α -optimal family $(f_R)_{R>0}$ always exists. Moreover we can assume w.l.o.g. that $\text{card}(f_R(E)) < \infty$ for every $R > 0$. Let us justify this. First, we note that $D_{\mu,r}^\alpha(R) > 0$ for every $R > 0$, because μ is a non-degenerate Gaussian measure. Hence, the left hand side of (7) is well-defined. Clearly, for every $R \geq 0$ we can define a quantizer $f \in \mathcal{F}$ with $H_\mu^\alpha(f) = 0 \leq R$*

(choose $a \in E$ and let $f(x) = a$ for every $x \in E$). Thus for every $R \geq 0$ a sequence $(f_R^n)_{n \in \mathbb{N}}$ of quantizers exists with $H_\mu^\alpha(f_R^n) \leq R$ and $D_{\mu,r}(f_R^n) \rightarrow D_{\mu,r}^\alpha(R)$ as $n \rightarrow \infty$. For every $R > 0$ choose $n_0(R)$ such that

$$|D_{\mu,r}(f_R^{n_0(R)}) - D_{\mu,r}^\alpha(R)| \leq \varepsilon_R \text{ with } \varepsilon_R = R^{-1} D_{\mu,r}^\alpha(R).$$

Consequently, $(f_R^{n_0(R)})_{R>0}$ is an asymptotically α -optimal family, i.e. such a family always exists.

Now let $(f_R)_{R>0}$ be an asymptotically α -optimal family, let ε_R as above and $a_0 \in f_R(E)$. Choose $A \subset f_R(E) \setminus \{a_0\}$ such that $\text{card}(f_R(E) \setminus A) < \infty$,

$$\sum_{a \in A} \int_{f_R^{-1}(a)} \|x - a\|^r d\mu(x) \leq \varepsilon_R/2 \text{ and } \int_{\cup_{a \in A} f_R^{-1}(a)} \|x - a_0\|^r d\mu(x) \leq \varepsilon_R/2.$$

With the quantizer

$$g_R = a_0 1_{\cup_{a \in \{a_0\} \cup A} f_R^{-1}(a)}(\cdot) + \sum_{a \in f_R(E) \setminus (\{a_0\} \cup A)} a 1_{f_R^{-1}(a)}(\cdot)$$

we obtain

$$|D_{\mu,r}(f_R) - D_{\mu,r}(g_R)| \leq \varepsilon_R.$$

Moreover, $H_\mu^\alpha(g_R) \leq H_\mu^\alpha(f_R)$ according to [31, Proposition 4.2] and [31, Definition 2.1(b)]. Thus, $(g_R)_{R>0}$ is also asymptotically α -optimal with $\text{card}(g_R(E)) < \infty$ for every $R > 0$.

Lemma 3.8. Let $\alpha \in]1, \infty[$ and $(f_R)_{R>0} \subset \mathcal{F}$ be an asymptotically α -optimal family. Then, $\lim_{R \rightarrow \infty} H_\mu^\alpha(R) = \infty$, or, what is the same,

$$\lim_{R \rightarrow \infty} \left(\sum_{a \in f_R(E)} \mu(f_R^{-1}(a))^\alpha \right)^{1/\alpha} = 0.$$

Proof. Let $C > 0$ and $(R_n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} R_n = \infty$. We will show that $\liminf_{n \rightarrow \infty} H_\mu^\alpha(f_{R_n}) \geq C$. Let us assume the contrary. Hence, there exists a subsequence of $(f_{R_n})_{n \in \mathbb{N}}$, which we will also denote by $(f_{R_n})_{n \in \mathbb{N}}$, such that $H_\mu^\alpha(f_{R_n}) < C$ for every $n \in \mathbb{N}$. By definition (1) of the optimal quantization error and because the support of μ is infinite we obtain

$$\liminf_{n \rightarrow \infty} D_{\mu,r}(f_{R_n}) \geq D_{\mu,r}^\alpha(C) > 0. \quad (8)$$

Let $\varepsilon > 0$. Then there exists an $R_\varepsilon > 0$, such that

$$D_{\mu,r}(f_R) \leq (1 + \varepsilon) D_{\mu,r}^\alpha(R) \quad \text{for every } R \geq R_\varepsilon. \quad (9)$$

From (9) and Remark 3.5 we get

$$0 \leq \lim_{R \rightarrow \infty} D_{\mu,r}(f_R) \leq (1 + \varepsilon) \lim_{R \rightarrow \infty} D_{\mu,r}^\alpha(R) = 0,$$

which contradicts (8). Thus we obtain that $\liminf_{n \rightarrow \infty} H_\mu^\alpha(f_{R_n}) \geq C$. Because $C > 0$ and $(R_n)_{n \in \mathbb{N}}$ was arbitrary we get $\lim_{R \rightarrow \infty} H_\mu^\alpha(R) = \infty$. Now the assertion follows immediately from the definition of $H_\mu^\alpha(f_R)$. \square

Remark 3.9. *As an immediate consequence of Lemma 3.8 we obtain*

$$\lim_{R \rightarrow \infty} \max_{a \in f_R(E)} \mu(f_R^{-1}(a)) = 0.$$

for $\alpha \in]1, \infty[$ and every asymptotically α -optimal family $(f_R)_{R>0} \subset \mathcal{F}$.

Proposition 3.10. *If*

$$\lim_{s \rightarrow 0} \frac{\mu(B(0, \eta s))}{\mu(B(0, s))} = 0 \quad (10)$$

for every $\eta \in]0, 1[$, then

$$D_{\mu,r}^\infty(R) \sim (b_\mu^{-1}(R))^r e^{-R} \text{ as } R \rightarrow \infty.$$

Proof. Let $\eta \in]0, 1[$ and $s \in]0, \infty[$ with $s = b_\mu^{-1}(R)$, i.e. $\mu(B(0, s)) = e^{-R}$. Proposition 2.3 implies

$$\begin{aligned} \frac{D_{\mu,r}^\infty(R)}{(b_\mu^{-1}(R))^r e^{-R}} &= \frac{\int_{B(0,s)} \|x\|^r d\mu(x)}{\mu(B(0,s)) s^r} \geq \frac{1}{\mu(B(0,s)) s^r} \int_{B(0,s) \setminus B(0,\eta s)} \|x\|^r d\mu(x) \\ &\geq \eta^r \left(1 - \frac{\mu(B(0,\eta s))}{\mu(B(0,s))}\right). \end{aligned}$$

Because $\eta \in]0, 1[$ is arbitrary and by assumption (10) we obtain that

$$\liminf_{R \rightarrow \infty} \frac{D_{\mu,r}^\infty(R)}{(b_\mu^{-1}(R))^r e^{-R}} \geq 1. \quad (11)$$

From Corollary 2.4 we get

$$\limsup_{R \rightarrow \infty} \frac{D_{\mu,r}^\infty(R)}{(b_\mu^{-1}(R))^r e^{-R}} \leq 1,$$

which yields together with (11) the assertion. \square

In the sequel we will apply results from the theory of slowly varying functions (cf. [7, Definition 1.2.1]). Let us state first the exact definition of this notion.

Definition 3.11. *Let $z > 0$ and $f : [z, \infty[\rightarrow \mathbb{R}^+$ be a Borel-measurable mapping satisfying*

$$f(\lambda x)/f(x) \rightarrow 1 \text{ as } x \rightarrow \infty \quad \forall \lambda > 0;$$

then f is said to be slowly varying.

Corollary 3.12. *Let $c > 0$. Let $a > 0$ and $b \in \mathbb{R}$. If*

$$b_\mu(s) \sim c(1/s)^a (\log(1/s))^b \text{ as } s \rightarrow 0,$$

then

$$D_{\mu,r}^\infty(R) \sim (b_\mu^{-1}(R))^r e^{-R} \sim c^{1/a} a^{-b/a} R^{-1/a} (\log(R))^{b/a} e^{-R} \text{ as } R \rightarrow \infty.$$

Proof. From [7, p.16] we obtain that $[1, \infty[\ni x \rightarrow c(\log(x))^b$ is a slowly varying function. Let $\eta \in]0, 1[$. Applying [7, Theorem 1.4.1] we deduce from [7, Definition (1.4.3)] that

$$1 < \eta^{-a} = \lim_{x \rightarrow \infty} \frac{b_\mu(1/(\eta^{-1}x))}{b_\mu(1/x)} = \lim_{s \rightarrow 0} \frac{b_\mu(\eta s)}{b_\mu(s)}.$$

Thus we obtain

$$\lim_{s \rightarrow 0} \frac{\mu(B(0, \eta s))}{\mu(B(0, s))} = \lim_{s \rightarrow 0} e^{b_\mu(s)(1 - b_\mu(\eta s)/b_\mu(s))} = 0.$$

Hence the first part of the assertion follows from Proposition 3.10. The second part is a direct consequence of Lemma 3.3. \square

Before we can state and prove our main result (Theorem 3.15) we need two more technical lemmas.

Lemma 3.13. *Let $\alpha > 1$, $A > 0$ and $B \geq 0$. Let $f : [0, 1/e[\rightarrow \mathbb{R}$ with*

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x (\log(1/x))^{-A} (\log(\log(1/x)))^B, & \text{if } x \in]0, 1/e[. \end{cases}$$

Then, f is continuous on $[0, 1/e[$, continuously differentiable on $]0, 1/e[$ and monotone increasing on $[0, e^{-e}[$. Moreover an $x_0 \in]0, 1/e[$ exists, such that the mapping $]0, x_0[\ni x \rightarrow F(x) = x^{1-\alpha} f'(x)$ is monotone decreasing.

Proof. Let $z > 1$ and $g(z) = z f(1/z)$. Thus g is slowly varying (cf. [7, Examples p. 16]). Applying [7, Proposition 1.3.6 (v)] we obtain

$$\lim_{x \rightarrow 0} f(x) = \lim_{z \rightarrow \infty} (1/z)g(z) = 0.$$

Thus f is continuous on $[0, 1[$. Now let $x \in]0, 1[$ and $z = 1/x$. We calculate

$$x^{1-\alpha} f'(x) = z^{\alpha-1} h_1(z) \cdot h_2(z)$$

with

$$h_1(z) = (\log(\log(z)))^{B-1} (\log(z))^{-A-1}$$

and

$$h_2(z) = \log(\log(z)) \log(z) + \log(\log(z)) - 1.$$

Because $f'(x) = h_1(1/x)h_2(1/x) > 0$ for every $x \in]0, e^{-e}[$ we obtain that f is monotone increasing on $[0, e^{-e}[$. Obviously h_1 and $h_2 - 1$ are slowly varying (cf. [7, p. 16]). Due to $h_2(z) \rightarrow \infty$ as $z \rightarrow \infty$ we obtain that also h_2 is slowly varying (cf. [7, Definition p. 6]) and, therefore, by means of [7, Proposition 1.3.6 (iii)], that $h_1 \cdot h_2$ is slowly varying. From [7, Theorem 1.3.1] we deduce that $h_1 \cdot h_2$ is a normalized slowly varying function (cf. [7, Definition p. 15]). Now [7, Theorem 1.5.5] yields that $h_1 \cdot h_2$ is an element of the so-called Zygmund class. From the definition of the Zygmund class (cf. [7, Definition p. 24]) we deduce that a $z_0 > 0$ exists, such that $z \rightarrow z^{\alpha-1} h_1(z) \cdot h_2(z)$ is increasing on $[z_0, \infty[$. But then F is decreasing on $]0, x_0[$ with $x_0 = 1/z_0$, which finally proves the assertion. \square

Lemma 3.14. *Let $\alpha \in]1, \infty[$. Let $x_0 > 0$ and $f : [0, x_0] \rightarrow \mathbb{R}_0^+$ be continuous with $f(0) = 0$. If f is continuously differentiable on $]0, x_0[$ and the mapping $]0, x_0[\ni x \rightarrow F(x) = x^{1-\alpha} f'(x)$ is monotone decreasing, then*

$$f(x_0) \leq \inf \left\{ \sum_{i=1}^n f(x_i) : n \in \mathbb{N}, (x_1, \dots, x_n) \in [0, x_0]^n, \sum_{i=1}^n x_i = 1, \sum_{i=1}^n x_i^\alpha \geq x_0^\alpha \right\}.$$

Proof. For $x \in [0, x_0]$ let $g(x) = f(x^{1/\alpha})$. For $n \in \mathbb{N}$ let

$$A_{n,\alpha}(x_0) = \{(x_1, \dots, x_n) \in [0, x_0]^\alpha : \sum_{i=1}^n x_i \geq x_0^\alpha\}.$$

Thus we obtain

$$\begin{aligned} & \inf \left\{ \sum_{i=1}^n f(x_i) : n \in \mathbb{N}, (x_1, \dots, x_n) \in [0, x_0]^n, \sum_{i=1}^n x_i = 1, \sum_{i=1}^n x_i^\alpha \geq x_0^\alpha \right\} \\ & \geq \inf \left\{ \sum_{i=1}^n g(x_i) : n \in \mathbb{N}, (x_1, \dots, x_n) \in A_{n,\alpha}(x_0) \right\}. \end{aligned}$$

$A_{n,\alpha}(x_0)$ is a convex set for every $n \in \mathbb{N}$. Moreover g is differentiable on $]0, x_0^\alpha[$ and for $x \in]0, x_0^\alpha[$ we calculate $g'(x^\alpha) = \alpha^{-1} x^{1-\alpha} f'(x)$. Thus g' is monotone decreasing on $]0, x_0^\alpha[$ and, therefore, concave on $]0, x_0^\alpha[$. For $n \in \mathbb{N}$ and $x \in A_{n,\alpha}(x_0)$ let $G(x) = \sum_{i=1}^n g(x_i)$. Obviously G is continuous and concave on the convex compact set $A_{n,\alpha}(x_0)$. Applying [42, Theorem 3.4.7.] we obtain that G attains its global minimum at an extreme point of $A_{n,\alpha}(x_0)$. Note that the extreme points of $A_{n,\alpha}(x_0)$ are consisting of the set

$$\{(x_1, \dots, x_n) \in [0, x_0^\alpha]^n : x_i \in \{0, x_0^\alpha\} \text{ for every } i \in \{1, \dots, n\} \setminus \{0\}\}.$$

Thus we get

$$\inf \left\{ \sum_{i=1}^n g(x_i) : n \in \mathbb{N}, (x_1, \dots, x_n) \in A_{n,\alpha}(x_0) \right\} \geq g(x_0^\alpha) = f(x_0),$$

which yields the assertion. \square

Theorem 3.15. *Let $\alpha \in]1, \infty[$ and $c > 0$. Let $a > 0$ and $b \in \mathbb{R}$. If*

$$b_\mu(s) \sim c(1/s)^a (\log(1/s))^b \text{ as } s \rightarrow 0,$$

then

$$D_{\mu,r}^\alpha(R) \sim \left(b_\mu^{-1} \left(\frac{\alpha-1}{\alpha} R \right) \right)^r e^{-\frac{\alpha-1}{\alpha} R} \sim D_{\mu,r}^\infty \left(\frac{\alpha-1}{\alpha} R \right)$$

as $R \rightarrow \infty$.

Proof. 1. $D_{\mu,r}^\alpha(R) \lesssim \left(b_\mu^{-1} \left(\frac{\alpha-1}{\alpha}R\right)\right)^r e^{-\frac{\alpha-1}{\alpha}R}$ as $R \rightarrow \infty$.

This follows from Lemma 3.1 and Corollary 3.12.

2. $D_{\mu,r}^\alpha(R) \gtrsim \left(b_\mu^{-1} \left(\frac{\alpha-1}{\alpha}R\right)\right)^r e^{-\frac{\alpha-1}{\alpha}R}$ as $R \rightarrow \infty$.

Let $(f_R)_{R>0}$ be an asymptotically α -optimal family of quantizers. According to Remark 3.7 let us assume w.l.o.g. that $\text{card}(f_R(E)) < \infty$ for every $R > 0$. By definition we have

$$D_{\mu,r}(f_R) = \sum_{a \in f_R(E)} \int_{f_R^{-1}(a)} \|x - a\|^r d\mu(x).$$

For every $a \in f_R(E)$ let $s_a(R) > 0$ such that $\mu(B(0, s_a)) = \mu(f_R^{-1}(a))$. Applying Lemma 2.2 and Proposition 2.3 we deduce

$$\int_{f_R^{-1}(a)} \|x - a\|^r d\mu(x) \geq \int_{B(0, s_a)} \|x\|^r d\mu(x) = D_{\mu,r}^\infty(-\log(\mu(f_R^{-1}(a)))).$$

Now let $\varepsilon \in]0, 1[$. According to Corollary 3.12 a $\delta \in]0, 1[$ exists, such that for every $x \in]0, \delta[$

$$D_{\mu,r}^\infty(-\log(x)) \geq (1 - \varepsilon)C \cdot g(x)$$

with $C = c^{1/a} a^{-b/a}$ and $g(x) = x(-\log(x))^{-1/a} (\log(-\log(x)))^{b/a}$. From Remark 3.9 we get an $R_1 > 0$ such that for every $R \geq R_1$ and for every $a \in f_R(E)$ we have

$$D_{\mu,r}^\infty(-\log(\mu(f_R^{-1}(a)))) \geq (1 - \varepsilon)C \cdot g(\mu(f_R^{-1}(a)))$$

and, therefore,

$$D_{\mu,r}(f_R) \geq (1 - \varepsilon)C \sum_{a \in f_R(E)} g(\mu(f_R^{-1}(a))). \quad (12)$$

Applying Lemma 3.13 we obtain a $z_0 \in]0, \delta[$ such that g is monotone increasing on $[0, z_0[$ and the mapping $]0, z_0[\ni x \rightarrow x^{1-\alpha} g'(x)$ is monotone decreasing. Let $x_0 = x_0(R, f_R) = (\sum_{a \in f_R(E)} \mu(f_R^{-1}(a))^\alpha)^{1/\alpha}$ and choose according to Lemma 3.8 an $R_2 > 0$ such that $x_0 < z_0$ for every $R \geq R_2$. Now let $R \geq \max(R_1, R_2)$. From $H_\mu^\alpha(f_R) \leq R$ we obtain $z_0 > x_0 \geq e^{-\frac{\alpha-1}{\alpha}R}$. Hence we can apply Lemma 3.14 and deduce together with the monotonicity of g that

$$\sum_{a \in f_R(E)} g(\mu(f_R^{-1}(a))) \geq g(x_0) \geq g(e^{-\frac{\alpha-1}{\alpha}R}). \quad (13)$$

Combining (12) and (13) we obtain from Corollary 3.12 that

$$1 - \varepsilon \leq \liminf_{R \rightarrow \infty} \frac{D_{\mu,r}(f_R)}{C g(e^{-\frac{\alpha-1}{\alpha}R})} = \liminf_{R \rightarrow \infty} \frac{D_{\mu,r}^\alpha(R)}{\left(b_\mu^{-1} \left(\frac{\alpha-1}{\alpha}R\right)\right)^r e^{-\frac{\alpha-1}{\alpha}R}}.$$

Because ε was arbitrary this proves the assertion of step 2.

3. $D_{\mu,r}^\infty \left(\frac{\alpha-1}{\alpha}R\right) \sim \left(b_\mu^{-1} \left(\frac{\alpha-1}{\alpha}R\right)\right)^r e^{-\frac{\alpha-1}{\alpha}R}$ as $R \rightarrow \infty$.

This follows from Corollary 3.12. □

Corollary 3.16. *Let $\alpha \in]1, \infty[$ and $c > 0$. Let $a > 0$ and $b \in \mathbb{R}$. If*

$$b_\mu(s) \sim c(1/s)^a (\log(1/s))^b \text{ as } s \rightarrow 0,$$

then

$$D_{\mu,r}^\alpha(R) \sim c^{1/a} a^{-b/a} \left(\frac{\alpha-1}{\alpha} R\right)^{-1/a} (\log(\frac{\alpha-1}{\alpha} R))^{b/a} e^{-\frac{\alpha-1}{\alpha} R}$$

as $R \rightarrow \infty$.

Proof. Immediate consequence of Theorem 3.15 and Corollary 3.12. \square

Remark 3.17. *Unfortunately the author was unable to prove the assertion of Theorem 3.15 under the weaker condition (10) of Proposition 3.10. In general it would be of interest to characterize those Gaussian measures who are satisfying condition (10). Finally it is also an open question if the asymptotics in Theorem 3.15 resp. Corollary 3.12 remain valid in a weak sense if we only require weak asymptotics for the small ball function, i.e. if we replace \sim by \approx .*

4 Examples

Let $d \in \mathbb{N}$. Let $f : [0, 1]^d \rightarrow \mathbb{R}$. We denote by $\|\cdot\|_\infty$ the sup-norm, i.e.

$$\|f\|_\infty = \sup_{t \in [0,1]^d} |f(t)|.$$

Moreover, for $p \geq 1$ and p -integrable mapping f let

$$\|f\|_p = \left(\int_{[0,1]^d} |f|^p dt \right)^{1/p}$$

be the L_p -norm of f . In the sequel we consider centered Gaussian probability vectors $X = (X_t)_{t \in [0,1]^d}$ on the separable Banach space $(C, \|\cdot\|_\infty)$ of all continuous functions on $[0, 1]^d$ and on the separable Banach space $(L_p, \|\cdot\|_p)$ of all p -integrable functions, respectively.

To be more precise we write $D_{\mu,r,\|\cdot\|_\infty}^\alpha(\cdot)$ resp. $D_{\mu,r,\|\cdot\|_p}^\alpha(\cdot)$ for the optimal quantization error in order to stress the dependency on the underlying norm and norm exponent $r > 0$. Moreover we write $b_{\mu,\|\cdot\|_\infty}(s)$ resp. $b_{\mu,\|\cdot\|_p}(s)$ for the small ball function.

Although results for small ball probabilities of Gaussian measures are also available for other norms (see e.g. [35]) we restrict ourself to the two norms from above.

4.1 Fractional Brownian sheet

Let $H = (H_1, \dots, H_d) \in]0, 1[^d$. Consider the centered Gaussian probability vector

$$X^H = (X_t^H)_{t \in [0,1]^d}$$

characterized by the covariance function

$$\mathbb{E}X_t^H X_s^H = \prod_{i=1}^d \frac{s_i^{2H_i} + t_i^{2H_i} - |s_i - t_i|^{2H_i}}{2},$$

with $s, t \in [0, 1]^d$. Fractional Brownian motion is covered by the special case $d = 1$. Moreover we obtain the classical Brownian sheet by letting $d = 2$ and $H_1 = H_2 = 1/2$. Let $\gamma = \min(H_1, \dots, H_d) > 0$.

Case 1. there is a unique minimum among H_1, \dots, H_d .

In this case we know, that a $c = c(H) \in]0, \infty[$ exists, such that

$$b_{\mu, \|\cdot\|_\infty}(s) \sim cs^{-1/\gamma} \text{ as } s \rightarrow 0$$

(cf. [39]. See also [33] and [34] for $d = 1$). From Lemma 3.3 we deduce

$$b_{\mu, \|\cdot\|_\infty}^{-1}(R) \sim (R/c)^{-\gamma} \text{ as } R \rightarrow \infty.$$

Corollary 3.12 implies

$$D_{\mu, r, \|\cdot\|_\infty}^\infty(R) \sim \left(\frac{c}{R}\right)^{\gamma r} e^{-R} \text{ as } R \rightarrow \infty.$$

From [25, Corollary 1.3] we deduce

$$D_{\mu, r, \|\cdot\|_\infty}^0(R) \approx \left(\frac{1}{R}\right)^{\gamma r} \text{ as } R \rightarrow \infty.$$

In one dimension ($d = 1$) we know (cf. relation (3.2) in [25]) that

$$b_{\mu, \|\cdot\|_p}(s) \approx s^{-1/\gamma} \text{ as } s \rightarrow 0$$

and (cf. [25, Corollary 1.3])

$$D_{\mu, r, \|\cdot\|_p}^0(R) \approx \left(\frac{1}{R}\right)^{\gamma r} \text{ as } R \rightarrow \infty.$$

Applying Remark 3.4 and Corollary 2.4 we obtain

$$D_{\mu, r, \|\cdot\|_p}^\infty(R) \lesssim \left(\frac{1}{R}\right)^{\gamma r} e^{-R} \text{ as } R \rightarrow \infty.$$

Sharp asymptotics for $b_{\mu, \|\cdot\|_p}(\cdot)$ are known if $p = 2$ (cf. [9]). Thus a $c_2 > 0$ exists such that

$$b_{\mu, \|\cdot\|_2}(s) \sim c_2 s^{-1/\gamma} \text{ as } s \rightarrow 0.$$

As above, Lemma 3.3 and Corollary 3.12 yields

$$D_{\mu, r, \|\cdot\|_2}^\infty(R) \sim \left(\frac{c_2}{R}\right)^{\gamma r} e^{-R} \text{ as } R \rightarrow \infty.$$

For $\alpha \in \{0, 1\}$ and $d = 1$ we have (cf. [14, Theorem 1.1.])

$$D_{\mu, r, \|\cdot\|_\infty}^\alpha(R) \sim \left(\frac{c_0(\gamma)}{R} \right)^{\gamma r} \text{ as } R \rightarrow \infty \quad (14)$$

for some constant $c_0(\gamma) \in]0, \infty[$. In view of Remark 1.2 relation (14) is also true for all $\alpha \in]0, 1[$. Moreover a $c_p(\gamma)$ exists such that

$$D_{\mu, r, \|\cdot\|_p}^\alpha(R) \sim \left(\frac{c_p(\gamma)}{R} \right)^{\gamma r} \text{ as } R \rightarrow \infty$$

for every $\alpha \in [0, 1]$ (cf. [14, Theorem 1.3.]). Independent of the norm of the Banach space E , the asymptotical order of $D_{\mu, r}^\alpha(R)$ for large R remains constant for $\alpha \in [0, 1]$. If $\alpha \in]1, \infty[$, then the asymptotical order changes. The asymptotic can be determined by applying Corollary 3.16. We obtain

$$D_{\mu, r, \|\cdot\|_\infty}^\alpha(R) \sim \left(\frac{\alpha}{\alpha - 1} \cdot \frac{c}{R} \right)^{\gamma r} e^{-\frac{\alpha-1}{\alpha} R} \text{ as } R \rightarrow \infty$$

and

$$D_{\mu, r, \|\cdot\|_2}^\alpha(R) \sim \left(\frac{\alpha}{\alpha - 1} \cdot \frac{c_2}{R} \right)^{\gamma r} e^{-\frac{\alpha-1}{\alpha} R} \text{ as } R \rightarrow \infty$$

for every $\alpha \in]1, \infty[$.

Case 2: there is a non-unique minimum among H_1, \dots, H_d .

Because the one-dimensional case has been already treated in case 1 we can assume w.l.o.g. that $d \geq 2$. If $d \geq 3$, then the asymptotical order of $b_{\mu, \|\cdot\|_\infty}(\cdot)$ is not yet completely determined, even if all H_i are equal. (cf. [17] and the references therein). If $d = 2$, then $H_1 = H_2 = H$ and we have

$$b_{\mu, \|\cdot\|_\infty}(s) \approx (1/s)^{1/H} (\log(1/s))^{1+1/H} \text{ as } s \rightarrow 0$$

(cf. [5, Theorem 5.2.], see also [44] for the case $H = 1/2$). Remark 3.4 implies

$$b_{\mu, \|\cdot\|_\infty}^{-1}(R) \approx R^{-H} (\log(R))^{H+1} \text{ as } R \rightarrow \infty.$$

Corollary 2.4 yields

$$D_{\mu, r, \|\cdot\|_\infty}^\infty(R) \lesssim (R^{-H} (\log(R))^{H+1})^r e^{-R} \text{ as } R \rightarrow \infty.$$

From [25, Corollary 1.3] we know that

$$D_{\mu, r, \|\cdot\|_\infty}^0(R) \approx (R^{-H} (\log(R))^{H+1})^r \text{ as } R \rightarrow \infty.$$

If $d \geq 2$ and $H_i = 1/2$ for every $i = 1, \dots, d$ we know (cf. [11]) that

$$b_{\mu, \|\cdot\|_2}(s) \sim c_d (1/s)^2 (\log(1/s))^{2d-2} \text{ as } s \rightarrow 0$$

with $c_d = 2^{d-2} / (\sqrt{2} \pi^{d-1} (d-1)!)$. Lemma 3.3 implies

$$b_{\mu, \|\cdot\|_2}^{-1}(R) \sim c_d^{1/2} 2^{-(d-1)} R^{-1/2} (\log(R))^{d-1} \text{ as } R \rightarrow \infty.$$

Corollary 3.12 yields

$$D_{\mu,r,\|\cdot\|_2}^\infty(R) \sim (c_d^{1/2} 2^{-(d-1)} R^{-1/2} (\log(R))^{d-1})^r e^{-R} \text{ as } R \rightarrow \infty.$$

From [25, Corollary 1.3] we obtain

$$D_{\mu,r,\|\cdot\|_2}^0(R) \approx (R^{-1/2} (\log(R))^{d-1})^r \text{ as } R \rightarrow \infty.$$

Moreover we know (cf. [36], relation (3.13)) that

$$D_{\mu,2,\|\cdot\|_2}^\alpha(R) \sim (b_d R^{-1/2} (\log(R))^{d-1})^2 \text{ as } R \rightarrow \infty \quad (15)$$

with $b_d = \sqrt{2}/(\pi^d(d-1)!)$ and $\alpha = 0$. From [36, Theorem 2.2.] and [26, Theorem 1.1.] we deduce that (15) also holds for $\alpha = 1$. In view of Remark 1.2 relation (15) is also true for all $\alpha \in]0, 1[$. As in Case 1 the asymptotical order of $D_{\mu,r}^\alpha(R)$ for large R and $\alpha > 1$ is different from the one for $\alpha \in [0, 1]$. It can be determined by applying Corollary 3.16. We get

$$D_{\mu,r,\|\cdot\|_2}^\alpha(R) \sim \left(c_d^{1/2} 2^{-(d-1)} \left(\frac{\alpha-1}{\alpha} R \right)^{-1/2} \left(\log \left(\frac{\alpha-1}{\alpha} R \right) \right)^{d-1} \right)^r e^{-\frac{\alpha-1}{\alpha} R}$$

as $R \rightarrow \infty$ for every $\alpha \in]1, \infty[$.

4.2 Lévy fractional Brownian motion

The Lévy fractional Brownian motion of order $H \in]0, 1[$ is a centered Gaussian process defined by

$$X_0 = 0, \quad \mathbb{E}((X_t - X_s)^2) = \|t - s\|^{2H} \text{ for } s, t \in [0, 1]^d,$$

if $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d . For this stochastic process we have

$$b_{\mu,\|\cdot\|_\infty}(s) \approx (1/s)^{d/H} \text{ as } s \rightarrow 0$$

(cf. [43]). Remark 3.4 yields

$$b_{\mu,\|\cdot\|_\infty}^{-1}(R) \approx (1/R)^{H/d} \text{ as } R \rightarrow \infty.$$

For $\alpha > 1$ Corollary 2.4 and Lemma 3.1 implies

$$D_{\mu,r,\|\cdot\|_\infty}^\alpha \left(\frac{\alpha}{\alpha-1} R \right) \leq D_{\mu,r,\|\cdot\|_\infty}^\infty(R) \lesssim (R^{-H/d})^r e^{-R} \text{ as } R \rightarrow \infty.$$

Applying [25, Corollary 1.3] we obtain

$$D_{\mu,r,\|\cdot\|_\infty}^0(R) \approx (R^{-H/d})^r \text{ as } R \rightarrow \infty.$$

4.3 m -times integrated Brownian Motion, Fractional Integrated Brownian Motions, m -integrated Brownian sheet

For $\beta > 0$ we define the centered Gaussian probability vector

$$X^\beta = (X_t^\beta)_{t \in [0,1]}$$

by

$$X_t^\beta = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} B_s ds, \quad t \in [0,1],$$

where B_s denotes Brownian motion. Since a $c(\beta) \in]0, \infty[$ exists, such that

$$b_{\mu, \|\cdot\|_\infty}(s) \sim c(\beta) s^{-2/(2\beta+1)} \text{ as } s \rightarrow 0$$

(cf. [33] and [34]) we deduce from Lemma 3.3 that

$$b_{\mu, \|\cdot\|_\infty}^{-1}(R) \sim (R/c(\beta))^{-(\beta+1/2)} \text{ as } R \rightarrow \infty.$$

Together with Corollary 3.12 we obtain

$$D_{\mu, r, \|\cdot\|_\infty}^\infty(R) \sim \left(\frac{c(\beta)}{R} \right)^{(\beta+1/2)r} e^{-R} \text{ as } R \rightarrow \infty.$$

Corollary 3.16 yields

$$D_{\mu, r, \|\cdot\|_\infty}^\alpha(R) \sim \left(\frac{\alpha}{\alpha-1} \cdot \frac{c(\beta)}{R} \right)^{(\beta+1/2)r} e^{-\frac{\alpha-1}{\alpha}R} \text{ as } R \rightarrow \infty$$

for every $\alpha \in]1, \infty[$. Moreover we have

$$D_{\mu, r, \|\cdot\|_\infty}^0(R) \approx \left(\frac{1}{R} \right)^{(\beta+1/2)r} \text{ as } R \rightarrow \infty.$$

(cf. [25, p. 1059]). If $\beta = m \in \mathbb{N}$, then a $c(m) > 0$ exists, such that

$$b_{\mu, \|\cdot\|_2}(s) \sim c(m) s^{-2/(2m+1)} \text{ as } s \rightarrow 0$$

(cf. [10, Theorem 1.1]). Again, Lemma 3.3 and Corollary 3.12 are implying that

$$D_{\mu, r, \|\cdot\|_2}^\infty(R) \sim \left(\frac{c(m)}{R} \right)^{(m+1/2)r} e^{-R} \text{ as } R \rightarrow \infty.$$

Applying [25, Corollary 1.3] we deduce

$$D_{\mu, r, \|\cdot\|_2}^0(R) \approx \left(\frac{1}{R} \right)^{(m+1/2)r} \text{ as } R \rightarrow \infty.$$

Moreover we know (cf. [36], relation (3.7)) that

$$D_{\mu,2,\|\cdot\|_2}^\alpha(R) \sim \left(\frac{c_0(m)}{R}\right)^{(m+1/2)r} \text{ as } R \rightarrow \infty \quad (16)$$

with a $c_0(m) \in]0, \infty[$ and $\alpha = 0$. From [36, Theorem 2.2.] and [26, Theorem 1.1.] we deduce that (16) also holds for $\alpha = 1$. In view of Remark 1.2 relation (16) is also true for all $\alpha \in]0, 1[$. If $\alpha \in]1, \infty[$, then Corollary 3.16 yields the error asymptotics. We deduce

$$D_{\mu,r,\|\cdot\|_2}^\alpha(R) \sim \left(\frac{\alpha}{\alpha-1} \frac{c(m)}{R}\right)^{(m+1/2)r} e^{-\frac{\alpha-1}{\alpha}R} \text{ as } R \rightarrow \infty$$

for every $\alpha \in]1, \infty[$. Results for small ball asymptotics of more general m -times integrated Brownian motions can be found in [22] and [41].

Now let $m \in \mathbb{N}$ and $(B_t)_{t \in [0,1]^d}$ be a d -dimensional Brownian sheet, i.e. (B_t) is a centered Gaussian measure characterized through the covariance function

$$\mathbb{E}(B_s B_t) = \prod_{j=1}^d \min(s_j, t_j)$$

for $s = (s_1, \dots, s_d) \in [0, 1]^d$ and $t = (t_1, \dots, t_d) \in [0, 1]^d$. The m -integrated Brownian sheet $(X_t)_{t \in \mathbb{R}^d}$ is now defined by

$$X_m(t) = \int_0^{t_1} \dots \int_0^{t_d} \prod_{j=1}^d \frac{(t_j - u_j)^m}{m!} B(du_1, \dots, du_d).$$

For this process a $c = c(m, d) > 0$ exists, such that

$$b_{\mu,\|\cdot\|_2}(s) \sim c s^{-2/(2m+1)} (\log(1/s))^{\frac{(d-1)(2m+2)}{2m+1}} \text{ as } s \rightarrow 0$$

(cf. [21, Corollary 5.2]). Lemma 3.3 yields

$$b_{\mu,\|\cdot\|_2}^{-1}(R) \sim \left(\frac{c}{R}\right)^{m+1/2} \left(\frac{\log(R)}{m+1/2}\right)^{(d-1)(m+1)} \text{ as } R \rightarrow \infty.$$

Corollary 3.12 implies

$$D_{\mu,r,\|\cdot\|_2}^\infty(R) \sim \left(\left(\frac{c}{R}\right)^{m+1/2} \left(\frac{\log(R)}{m+1/2}\right)^{(d-1)(m+1)}\right)^r e^{-R} \text{ as } R \rightarrow \infty.$$

On the other hand we deduce from [25, Corollary 1.3] that

$$D_{\mu,r,\|\cdot\|_2}^0(R) \approx \left(R^{-(m+1/2)} (\log(R))^{(d-1)(m+1)}\right)^r \text{ as } R \rightarrow \infty.$$

In case of $\alpha > 1$ we obtain sharp asymptotics for the optimal quantization error by Corollary 3.16. We have

$$D_{\mu,r,\|\cdot\|_2}^\alpha(R) \sim \left(\left(\frac{\alpha}{\alpha-1} \frac{c}{R}\right)^{m+1/2} \left(\frac{\log(\frac{\alpha-1}{\alpha}R)}{m+1/2}\right)^{(d-1)(m+1)}\right)^r e^{-\frac{\alpha-1}{\alpha}R}$$

as $R \rightarrow \infty$ for every $\alpha \in]1, \infty[$.

4.4 Fractional Ornstein-Uhlenbeck Processes

Let $\gamma > 0$ and $H \in]0, 2[$. Let us consider the centered stationary fractional Ornstein-Uhlenbeck process, which is a Gaussian process defined by the covariance function

$$\mathbb{E}X_t^H X_s^H = e^{-\gamma|t-s|^H}, \quad t, s \in [0, 1].$$

Since

$$b_{\mu, \|\cdot\|_\infty}(s) \approx s^{-2/H} \text{ as } s \rightarrow 0$$

(cf. [40, Theorem 2.1]) and

$$b_{\mu, \|\cdot\|_p}(s) \approx s^{-2/H} \text{ as } s \rightarrow 0$$

(cf. [25, p. 1061]) we deduce from Remark 3.4 and Corollary 2.4 together with Lemma 3.1 for $\alpha > 1$ that

$$D_{\mu, \|\cdot\|_\infty}^\alpha\left(\frac{\alpha}{\alpha-1}R\right) \leq D_{\mu, \|\cdot\|_\infty}^\infty(R) \lesssim \left(R^{-H/2}\right)^r e^{-R} \text{ as } R \rightarrow \infty$$

and

$$D_{\mu, \|\cdot\|_p}^\alpha\left(\frac{\alpha}{\alpha-1}R\right) \leq D_{\mu, \|\cdot\|_p}^\infty(R) \lesssim \left(R^{-H/2}\right)^r e^{-R} \text{ as } R \rightarrow \infty.$$

Moreover (cf. [25])

$$D_{\mu, \|\cdot\|_\infty}^0(R) \approx \left(R^{-H/2}\right)^r \text{ as } R \rightarrow \infty.$$

and

$$D_{\mu, \|\cdot\|_p}^0(R) \approx \left(R^{-H/2}\right)^r \text{ as } R \rightarrow \infty.$$

If $H = 1$, then we have the standard Ornstein-Uhlenbeck process which can also be defined as the solution of a stochastic differential equation. From this special case we can also generalize the standard Ornstein-Uhlenbeck process to Gaussian diffusions, defined as a solution of a certain stochastic differential equation. Asymptotic small ball probabilities for such processes were derived by Fatalov [20]. For results about the asymptotics of the optimal quantization error for such diffusions and $\alpha \in \{0, 1\}$ the reader is referred to Dereich [15, 16] resp. Luschgy and Pagès [37, 38]. The optimal quantization of Fractional Ornstein-Uhlenbeck Processes with higher dimensional index space has been discussed by Luschgy and Pagès [36]. Once again we observe the change in the asymptotical order of the high rate asymptotics of the optimal quantization error, if the entropy index α becomes larger than 1.

Remark 4.1. *Taking the sum $Z = X + Y$ of two not necessarily independent joint Gaussian random vectors X, Y it is possible to determine the asymptotical order of the small ball probability of Z , if this order is known for X and Y (cf. [19, Theorem 2.1]). Moreover, small ball probabilities of fractional mixtures of fractional Gaussian measures are investigated by El-Nouty [18, 19].*

4.5 Slepian Gaussian fields

Let $a = (a_1, \dots, a_d) \in]0, \infty[^d$. We consider the centered Gaussian process $(X_t)_{t \in [0,1]^d}$ characterized by the covariance function

$$\mathbb{E}(X_t X_s) = \prod_{i=1}^d \max(0, a_i - |s_i - t_i|).$$

For this process we have

$$b_{\mu, \|\cdot\|_\infty}(s) \approx (1/s)^2 (\log(1/s))^3 \text{ as } s \rightarrow 0$$

(cf. [23, Theorem 1.1]) and

$$b_{\mu, \|\cdot\|_2}(s) \approx (1/s)^2 (\log(1/s))^{2d-2} \text{ as } s \rightarrow 0$$

(cf. [23, Theorem 1.1]). Thus we obtain from Remark 3.4, Corollary 2.4 together with Lemma 3.1 for $\alpha > 1$ that

$$D_{\mu, r, \|\cdot\|_\infty}^\alpha \left(\frac{\alpha}{\alpha-1} R \right) \leq D_{\mu, r, \|\cdot\|_\infty}^\infty(R) \lesssim \left(R^{-1/2} (\log(R))^{3/2} \right)^r e^{-R} \text{ as } R \rightarrow \infty$$

and

$$D_{\mu, r, \|\cdot\|_2}^\alpha \left(\frac{\alpha}{\alpha-1} R \right) \leq D_{\mu, r, \|\cdot\|_2}^\infty(R) \lesssim \left(R^{-1/2} (\log(R))^{d-1} \right)^r e^{-R} \text{ as } R \rightarrow \infty.$$

Moreover (cf. [25])

$$D_{\mu, r, \|\cdot\|_\infty}^0(R) \approx \left(R^{-1/2} (\log(R))^{3/2} \right)^r \text{ as } R \rightarrow \infty$$

and

$$D_{\mu, r, \|\cdot\|_2}^0(R) \approx \left(R^{-1/2} (\log(R))^{d-1} \right)^r \text{ as } R \rightarrow \infty.$$

Finally, also this class of Gaussian processes shows the change in the optimal quantization error asymptotics as α increases.

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